

Expanding Dynamics

Def. A rational map is (dynamically) hyperbolic if the orbit of every critical point converges to an attracting cycle.

Ex: $f_c(z) = z^2 + c$ is hyperbolic when $\underbrace{f_c^n(0) \rightarrow c}$
or $\underbrace{f_c \text{ has an attr. cycle}}_{c \in \mathbb{C} \setminus M}$

Ex: $f(z) = 1 - 1/z^2$ $c \in M_{\text{hyp}} \subset M$

* f hyp $\Rightarrow$ no parabolic, Siegel, Malmgren, Cremona,
only attr. basin

Thm. f is hyperbolic iff $\exists$ open $U \supset J(f)$
st. $f^{-1}(U) \subset U$ and f is strongly expanding
on U with its hyp metric:

$$\exists K > 1 \text{ st. } \|Df\|_x \geq K \text{ if } x \in f^{-1}(U)$$

Proof.

f hyp $\Rightarrow$ expanding: Around each attracting point, take a Koenig neighborhood D_p st.
 $\overline{f(D_p)} \subset D_{f(p)}$. Let $V_n = \overline{f^{-n}(\bigcup_p D_p)}$,

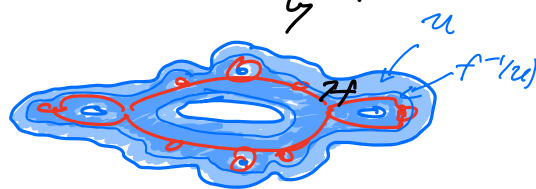
choose u so that $V_u \supset \overline{P}$.

Choose $U = (\hat{\mathbb{C}} \setminus V_u) \cap \{z \mid |f(z)| < 17\}$. Then $f^{-1}(u) \subset U$ by construction, $U \supset J$.

$\Rightarrow f^{-1}(u) \xrightarrow{\text{contr.}} U$, so on $f^{-1}(u)$, f^{-1} is approaching as the U -structure.
 $f^{-1}(u) \subset U \Rightarrow \text{struc. exp}$

conv. map $\rightarrow$
 back to f

f approaching $\Rightarrow$ hyp:



1) No end pts in $f^{-1}(u) \supset J$

2) If $z_0 \in U \cap F$, then the orbit count accumulates in U :

• γ path from z_0 to J ,

$$l(\gamma) \geq K \cdot l(f^{-1}(\gamma)) \geq K \cdot d(f^{-1}(z_0), J)$$

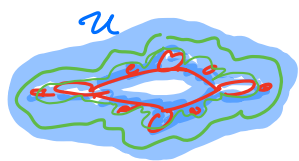
• $\text{lim } \gamma: d(z_0, J) \geq K \cdot d(f^{-1}(z_0), J)$.

$$\Rightarrow d(f^{-1}(z_0), J) \rightarrow 0.$$

3) Classification of Fatou $\rightarrow$ Crit $\subset$ Attr. Basins

Thm. If f is hyperbolic and $J(f)$ is connected, then $J(f)$ is loc. conn. Can HW

Proof for f poly: Will show that $\varphi: \hat{\mathbb{C}} \setminus D \rightarrow \hat{\mathbb{C}}^k$ extends to $\gamma: S^1 \rightarrow J(f)$:



Pick + st. $R_\theta(t) \in \mathcal{U}$.

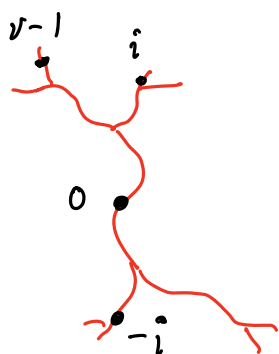
$$M = \max_{\theta \in S^1} d_u(R_\theta(u^{-1}t), R_\theta(t))$$

$$\Rightarrow d_{\text{Eucl.}}(R_\theta(u^{-1}t), R_\theta(u^{-(r-1)}t))$$

$$\leq d_u(R_\theta(u^{-1}t), R_\theta(u^{-(r-1)}t)) \leq K^{-1} M \rightarrow 0$$

$$\Rightarrow \{\theta \mapsto R_\theta(u^{-1}t)\} \text{ is unif. Cauchy}$$

Ex: $f_i(z) = z^2 + i$ PCF Misiurewicz



$$0 \mapsto i \mapsto i-1 \mapsto -i$$

Repelling cycle

$$\text{Ex: } f_{1/4}(z) = z^2 + \frac{1}{4}$$

Centifurcs



Def. 1) f is subhyperbolic if the orbit of every crit. point is either finite or converges to an attracting cycle

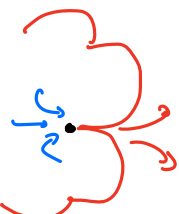
$$\text{Ex: } f_i$$

2) f is geometrically finite if the orbit of the crit. point are finite or converge to attracting or parabolic cycles

↳ No Angel, Helen, Anna Ex: $f_2, f_{1/4}$

Thm. If f is geometrically finite and $J(f)$ is connected, then $J(f)$ is locally connected.

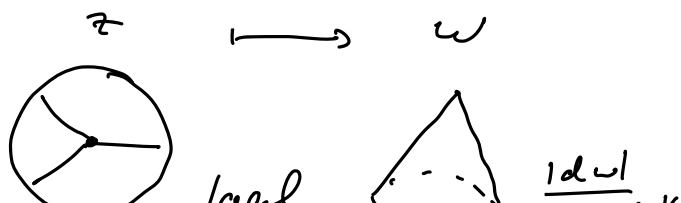
• cut $p \in J$: Can't be expanding at the
 cut. points.
 • If we remove P , not rel. compact,
 so not strongly expanding.

 • Not expanding in all directions at
 a parabolic.
 • Expansion is also not strong in the
 repelling direction.

• Constant norm geodesic metric, "geometry",
 analogy $\rightarrow$ hyperbolic 3-manifold

• Trick: $z \mapsto w = z^k$, push forward
 $dw = k z^{k-1} dz$
 the metric $|dz|$:

$$|dz| = \frac{|dw|}{k |w|^{1-1/k}}$$



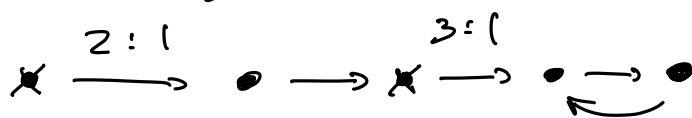
$|dz|$ isometry $\kappa |w|^{-1/\kappa}$
 - Singular at the center, but finite lengths

For $M \frac{|dw|}{\kappa \cdot |w|^{1/\kappa}}$, $\kappa > 1$, then it is expanding!

Proof sketch: A) Subhyperbolic:
 for poly

$$1) \text{Crit}(f) = C_{\text{attr}} \cup C_{\text{fin}}, P(f) = P_{\text{attr}} \cup P_{\text{fin}}$$

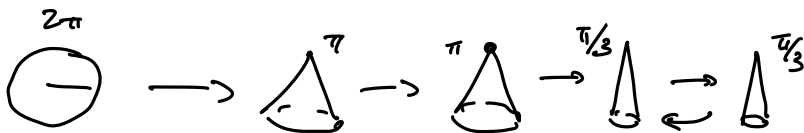
At any crit value, we want a cone:



$$v: \hat{\mathbb{C}} \rightarrow \mathbb{N}$$

$$v(z) = |\text{cm deg}_{z'} f^n|$$

$$f^n(z') = z$$



$$v: 1 \quad 2 \quad 2 \quad 6 \quad 6$$

$$v = 1 \text{ except on } P$$

$$(\deg_z f) \cdot v(p) \mid v(h(p))$$

2) Choose local coord $\mathbb{D}_p$ in a disk near p s.t.

$$f \text{ is contracting for the metrics } \mu_p = \frac{|d\mathbb{D}_p|}{|\mathbb{D}_p|^{1-1/v(p)}}$$

$$3) U = \hat{\mathbb{C}} \setminus \overline{P}$$

$$\mu = \inf \{ g_{\text{hyp}, u}, M \mu_p \}, M > 0$$



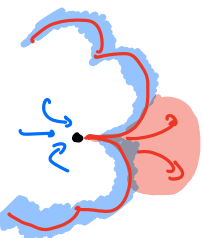
s.t. f is strongly exp.

4) Same very very converging exponential map

B)

$$1) U = \hat{\mathbb{C}} \setminus (P \cup \text{Attracting Periodic locus})$$

also repelling petals: Pick $\forall |dz|$ so that it is expanding.



2) Same as for subhyperbolic for other crit. points.

3) Piece together the pictures, get slower but still Carathéodory convergence.

Topological Model for Superattracting

$f(z) = z^2 + c$, superattracting per. $u \mapsto K(f)$ loc. conn.

$$A_0(0) \xrightarrow{f^n} A_0(0)$$

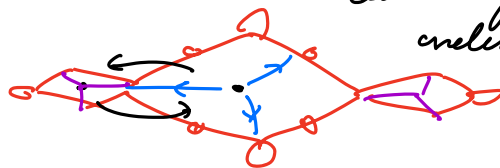
$$\psi \uparrow \quad \uparrow \text{Böttcher}$$

$$\mathbb{D} \longrightarrow \mathbb{D}$$

$$w_1 \longrightarrow w_2$$

internal rays $r \mapsto \varphi(re^{2\pi i t})$

and all preimages under f



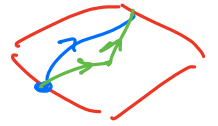
Def. An arc $\gamma \subset K$ is legal if $\gamma \cap K^0$ is the union of internal rays.

Lemma. 1) γ legal $\Rightarrow f(\gamma)$ is legal

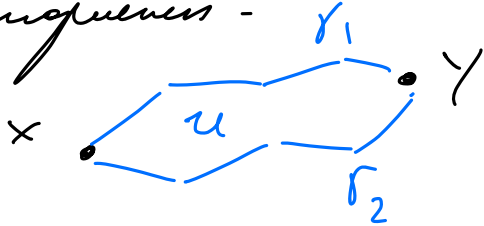
2) $\forall x, y \in K, \exists!$ legal arc $[x, y]$ from x to y

Proof. 2) Fact. Loc. connected compact metric spaces are path connected

$\gamma \subset K$, $\gamma: x \rightarrow y$, the decomposition of K° , chosen γ is
 internal rays. $\Rightarrow$ Legal



Uniqueness: $C \setminus K$ is connected $\Rightarrow U \subset K$
 γ_1 is a ray of $K^\circ \Rightarrow \gamma_1 = \gamma_2$

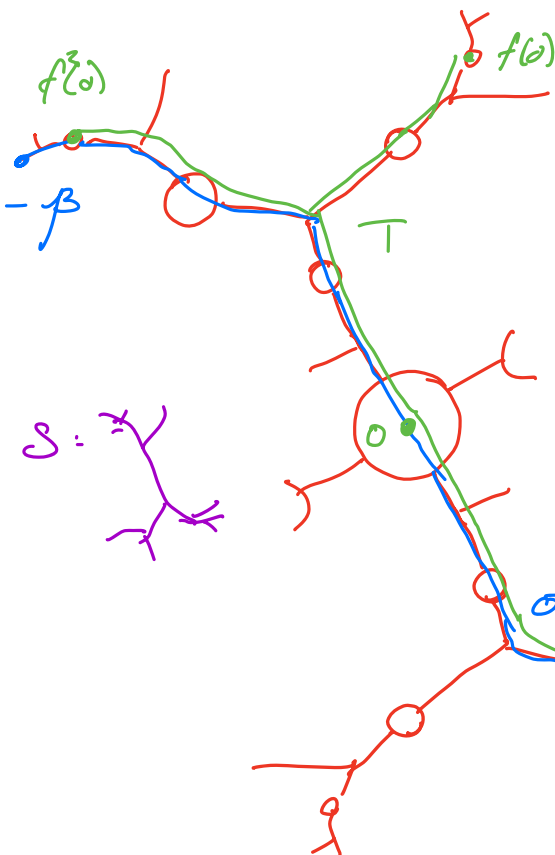


Def. $\beta := \gamma(0)$, landing point of R_0

Spine $\sigma = [\beta, -\beta]$, Skeleton $S = \bigcup_{n=0}^{\infty} f^{-n}(\sigma)$

Harmonized tree $T = \bigcup_{n,m} [f^n(0), f^m(0)]$

Ex: Moeckel's, $c \approx -0.156 + 1.032i$, $f_c^{-4}(0) = 0$



Lemma. 1) T is a tree, $f(T) = T$

2) $0 \in \sigma \cap T$

3) $c = f(0)$ is an extreme point of T

4) $[a, b] \cap K^\circ$ is dense in $[a, b]$

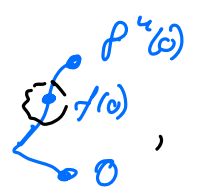
5) Every point of σ eventually enters $T \cap \sigma$

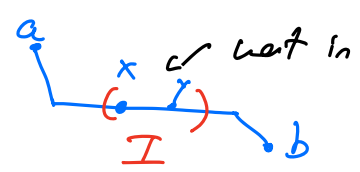
6) $S = \bigcup_{n \geq 0} f^{-n}(T)$

7) $S \cap J = \{ \text{cut points of } J \}$
 ≥ 2 external rays land

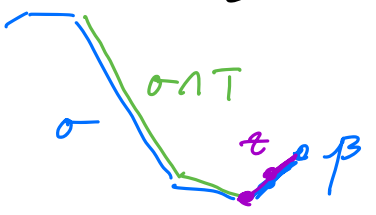
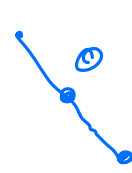
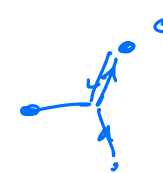
Proof. 1) $T \setminus \{0\} = T_1 \cup T_2$, fitting as each

2) Symmetry under $z \mapsto -z$.

3) Since $f'(f^u(0)) \neq 0$ if $u \neq 0 \pmod K$, if , local is $\Rightarrow$ no $f^u(0)$ in leftmost.

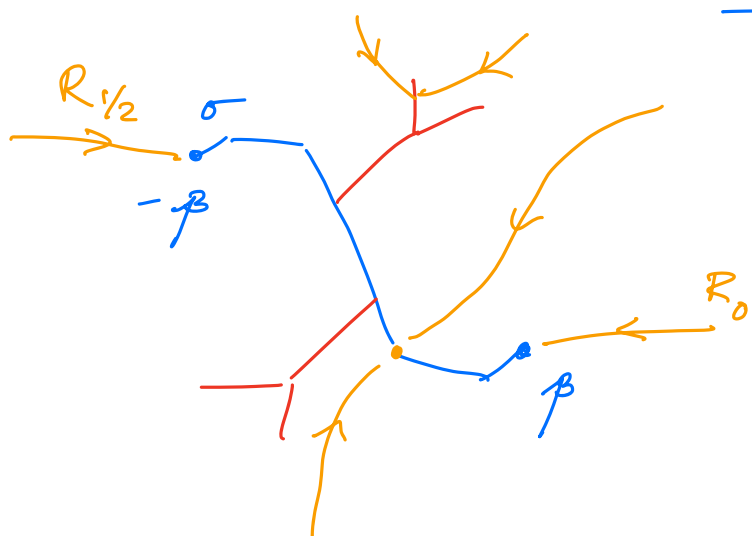
4) If not dense:  $\Rightarrow f^u(I) \neq A_0(0)$

Since f is hyp, $d(f^u(x), f^u(y)) \geq K^u d(x, y) \rightarrow \infty$, but the diameter is finite.

5)  $z \in \sigma$,  $\xrightarrow{f}$ , so apply f to $[\beta, z]$ until it gets stuck in T . Then it is in $[\beta, f^u(0)]$, up until is $[\beta, 0] \subset \sigma$. 

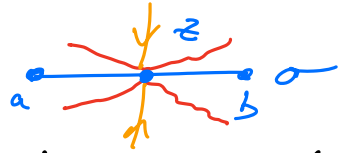
6) Construct from 5).

Claim

$\Rightarrow$ 

$\sigma \cap T = \{ \text{pts } w \text{ on external ray landing on each side of } \sigma \cup R_0 \cup R_{1/2} \}$

HW: A ray lands at $z \in J$ for each com.
component of $A(\infty) \cap \mathbb{D}_\varepsilon(z)$.



- z only maps on one side $\Rightarrow [a, z] \neq [b, z]$ lie in the same component of $K^\circ \Rightarrow a=b$.
- Then pull back to get S .

Def Root $r \in \partial A_0(c)$ is the landing point of the internal ray at angle 0 for $A_0(c)$.

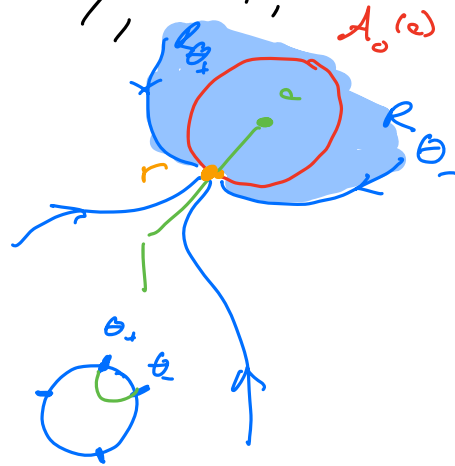


$\hookrightarrow$ The first part of $f^n: \overline{A_0(c)} \rightarrow \overline{A_1(c)}$

- Since c is a tip of T , $f^n(+)=T$, $\gamma_i \in T, S$

Characteristic angles $\theta_- < \theta_+$ are

the angles bounding the smallest interval in $S' \setminus \gamma^{-1}(\gamma_i)$. (Misiurewicz)



Lemma. c is the only $f^n(0)$ bounded by $R_{\theta_-} \neq R_{\theta_+}$

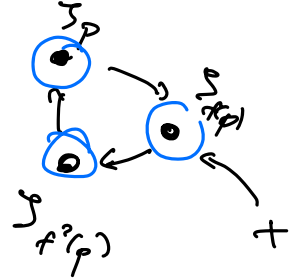
- $\gamma \in \mathcal{D}$ goes from between $\theta_- \neq \theta_+$, characteristic leaf

Theorem. $\mathcal{L} = \overline{\bigcup_{n \geq 0} E^{-n}(\gamma)}$ is the lamination of external rays for $K(f)$

Further details on the subtyp proof:

2.1) let σ click around each repelling periodic
 $p \in P_{fin}$, $f^k(p) = p$, λ mult, pick Koenig's esds
 Σ_p so that $f: \Sigma_p \rightarrow \lambda^{\frac{1}{k}} \Sigma_{f(p)}$. by nature

$$\mu_p = \frac{|d\Sigma_p|}{|\Sigma_p|^{1-\frac{1}{\nu(p)}}} \text{ it is expanding}$$



2.2) Pull back the metric: $\Sigma_p \xrightarrow{\lambda^{\frac{1}{k}}} \Sigma_{f(p)}$
 $\Sigma_p := r \Sigma_{f(p)}^{\frac{1}{k}}$ so $\mu_p = \frac{|d\Sigma_p|}{|\Sigma_p|^{1-\frac{1}{\nu(p)}}}$, f is expanding.